

Scale factor duality in quintessence models ?

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Abstract

We consider several kinds of quintessence models in the framework of scale factor duality. We show that this symmetry exists only for a very small number of quintessence potentials. We then apply the duality transformations found to several analytical solutions. It turns out that, in some cases, the presence of the potential allows a smooth connection between the pre- and the post-Big Bang phases. This may be a first step toward the resolution of the singularity *problem*.

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1 Introduction

Quintessence models have been proposed as an alternative to the cosmological constant in order to explain some recent astrophysical observations which seem to favour a currently accelerating universe [1]. In these models, a scalar field slowly rolling down its potential is added to the standard cosmological model. Depending on the way it is introduced, this new field can be interpreted as a fluid with a negative pressure or as a new ingredient to the gravitation theory. There are several reasons that lead one to favour a scalar field candidate in place of a cosmological constant. First of all, while the cosmological constant does not yet possess a completely satisfactory physical interpretation [2], the scalar field appears naturally in a large number of alternative theories to general relativity. Furthermore, in some of these alternative theories, the scalar terms play an important physical role and consequently cannot be neglected.

The effective string theory is one of those alternative theories. Its study in the cosmological framework has been motivated by its remarkable property of symmetry called *scale factor duality* (see [3] and references therein). This symmetry allows the construction of a *dual solution* from a known solution of the theory field equations. In this context, the initial solution is used to describe the “*post-Big Bang phase*”, i.e. the Universe evolution between the initial singularity until today, whereas the dual solution is associated to a possible “*pre-Big Bang phase*”, i.e. the Universe as it could have been before the Big Bang. Though there is no satisfying model explaining the transition between the pre- and the post-Big Bang phases, many cosmologists think that duality symmetry and in particular the pre-Big Bang phase could solve the singularity *problem*. It should be noted, however, that the string theory is not the only theory having this duality symmetry: many scalar-tensor theories have this property as well.

As one can see, scale factor duality and quintessence models are both very promising. However, only quintessence models without a scalar potential have been studied in the duality framework whereas the supernovae observations require a model with a potential in order to be interpreted by quintessence. The purpose of this paper is to see if the scale factor duality can be maintained in the presence of a scalar potential. The different kinds of cosmological models considered are those coming from the scalar-tensor theories with various couplings between the scalar field, gravitation and the matter source. For all these models, we determine which form the potential should have if the theory has to remain invariant under a scale factor duality symmetry. Finally, we apply the duality transformations found to some analytical solutions.

2 Non-minimally coupling with gravitation

The first kind of quintessence models we shall consider is the one coming from the effective string theory [3]. These models are the most studied in the framework of scale factor duality because duality has initially been developed in the context of the string theory.

The effective string action is given by

$$S = \frac{1}{2\kappa} \int e^{-\phi} [R + [\nabla\phi]^2 - V(\phi)] \sqrt{-g} d^4x + \int L_m \sqrt{-g} d^4x \quad (1)$$

where $\kappa = 8\pi G$ and where a scalar potential $V(\phi)$ has been introduced. In what follows, we shall only consider flat Friedmann-Lemaître-Robertson-Walker spacetimes so that the metric tensor can be written as:

$$g_{\alpha\beta} = \begin{pmatrix} g_{00}(t) & 0 \\ 0 & G(t) \end{pmatrix} \quad \text{with} \quad G(t) = \begin{pmatrix} a^2(t) & 0 & 0 \\ 0 & a^2(t) & 0 \\ 0 & 0 & a^2(t) \end{pmatrix} \quad (2)$$

When the scalar potential is absent, the duality transformation can be written as [3] [6]:

$$\begin{cases} G \rightarrow \bar{G} = G^{-1} \\ g_{00} \rightarrow \bar{g}_{00} = g_{00} \\ \phi \rightarrow \bar{\phi} = \phi - \ln(\det G) \\ \rho_m \rightarrow \bar{\rho}_m = [\det G] \rho_m \\ p_m \rightarrow \bar{p}_m = -[\det G] p_m \end{cases} \quad (3)$$

It remains to see which form the potential should have if we want the complete action (1) to remain invariant under duality. We also expect the potential to be the same function of the scalar field before and after the duality transformation. Following the transformation (3), we have $e^{-\phi} \sqrt{-g} \rightarrow e^{-\bar{\phi}} \sqrt{-\bar{g}}$ which implies that the potential must satisfy the following condition: $V(\phi) = \bar{V}(\bar{\phi}) = V(\bar{\phi})$. This is possible only if the potential is constant:

$$V(\phi) = \Lambda \rightarrow \bar{V}(\bar{\phi}) = \Lambda \quad (4)$$

where Λ is a constant. In the framework of the effective string theory, this constant can be interpreted as the *charge deficit* which is a constant depending on the compactification scheme used to bring the initial 10 dimensional action to the 4 dimensional action given by (1). Note that the case of an effective string theory containing a cosmological constant has already been discussed in the literature (see, e.g., [4]).

The equation of state for the matter field, i.e. $w_m = p_m / \rho_m$, remains invariant under the scale factor duality only for dust ($w_m = 0$). Furthermore, it has to be noted that the action (1) with $V(\phi) = \Lambda$ is equivalent to a Brans-Dicke theory with a linear potential:

$$S = \frac{1}{2\kappa} \int (\Phi R + \Phi^{-1} [\nabla\Phi]^2 - \Lambda \Phi) \sqrt{-g} d^4x \quad (5)$$

where $\Phi = e^{-\phi}$, or to a scalar-tensor theory with a power-law potential:

$$S = \frac{1}{2\kappa} \int (\varphi^2 R + 4 [\nabla\varphi]^2 - \Lambda \varphi^2) \sqrt{-g} d^4x \quad (6)$$

where $\varphi = e^{-\phi/2}$. Recently, the scale factor duality has been discussed in the framework of this scalar-tensor theory by de Ritis et al. in [7].

In the context of a flat FLRW spacetime ($g_{00} = -1$) filled with dust, the field equations deriving from the action (1) with $V(\phi) = \Lambda$ are given by

$$-6 \frac{a'^2}{a^2} + 6 \phi' \frac{a'}{a} - \phi'^2 + \Lambda + 2 e^\phi \kappa \rho_m = 0 \quad (7)$$

$$2 \phi'' + 4 \phi' \frac{a'}{a} - \phi'^2 - 4 \frac{a''}{a} - 2 \frac{a'^2}{a^2} + \Lambda = 0 \quad (8)$$

$$6 \frac{a''}{a} + 6 \frac{a'^2}{a^2} - 6 \phi' \frac{a'}{a} - 2 \phi'' + \phi'^2 - \Lambda = 0 \quad (9)$$

Meissner and Veneziano have already proposed a solution of this system when there is no source terms, i.e. when $p_m = \rho_m = 0$ [5]. We present here a solution in the presence of the matter field:

$$a(t) = a_0 e^{\alpha_0 \tau(t)} \quad (10)$$

$$\kappa \rho_m(t) = A a_0^{-3} e^{-3 \alpha_0 \tau(t)} \quad (11)$$

$$\phi(t) = 3 \alpha_0 \tau(t) + \ln \left(C_1 \left[\cosh[\sqrt{3} \alpha_0 \tau(t)] - \delta_0 \right] \right) \quad (12)$$

with the following constraints:

$$\Lambda = \frac{A^2}{3 \alpha_0^2} \left[1 - \frac{1}{\delta_0^2} \right] \quad C_1 = \frac{A a_0^3}{3 \alpha_0^2 \delta_0} \quad \delta_0 \leq 1 \quad (13)$$

The quantity τ present in (10)-(12) is called the *conformal time* and is defined by $d\tau = \pm a^{-3} e^\phi dt$. The function $\tau(t)$ depends on δ_0 :

$$\tau(t) = \frac{2}{\sqrt{3} \alpha_0} \operatorname{arctanh} \left[\pm \frac{a_0^3}{\sqrt{3} \alpha_0 C_1 t} \right] \quad \text{if } \delta_0 = 1 \quad (14)$$

$$\tau(t) = \frac{2}{\sqrt{3} \alpha_0} \operatorname{arctanh} \left(\frac{\sqrt{1 - \delta_0^2}}{1 + \delta_0} \tanh \left[\pm \frac{\sqrt{3(1 - \delta_0^2)} \alpha_0 C_1 t}{2 a_0^3} \right] \right) \quad \text{if } \delta_0 < 1 \quad (15)$$

The first relation in (13) shows that the case given by (14) is associated to a null cosmological constant whereas the case given by (15) corresponds to a negative cosmological constant. Furthermore, the solution without a cosmological constant does not exist for $|t| < a_0^3 / (\sqrt{3} C_1 \alpha_0)$.

If we apply transformation (3)-(4) to the solution given by (10)-(12), we obtain the following dual solution:

$$\bar{a}(t) = a_0^{-1} e^{-\alpha_0 \tau(t)} \quad (16)$$

$$\kappa \bar{\rho}_m(t) = A a_0^3 e^{3 \alpha_0 \tau(t)} \quad (17)$$

$$\bar{\phi}(t) = -6 \ln(a_0) - 3 \alpha_0 \tau(t) + \ln \left(C_1 \left[\cosh(\sqrt{3} \alpha_0 \tau(t)) - \delta_0 \right] \right) \quad (18)$$

where (13)-(15) are still valid. For $a_0 = 1$ and $\delta_0 < 1$, the solution and its dual can be joined smoothly without any singularity at $t = 0$. In order to clarify this interesting

feature, we plotted on the figure 1 the solution and its dual given by respectively (10)-(12) and (16)-(18) for $\delta_0 = 1$ and $\delta_0 = 0.5$. The figure 1 clearly shows that it is impossible to allow a smooth transition between the pre- and the post-Big Bang solutions without a cosmological constant. Notice that the pre-Big Bang has been obtained by making the time inversion $t \rightarrow -t$ in the dual solution.

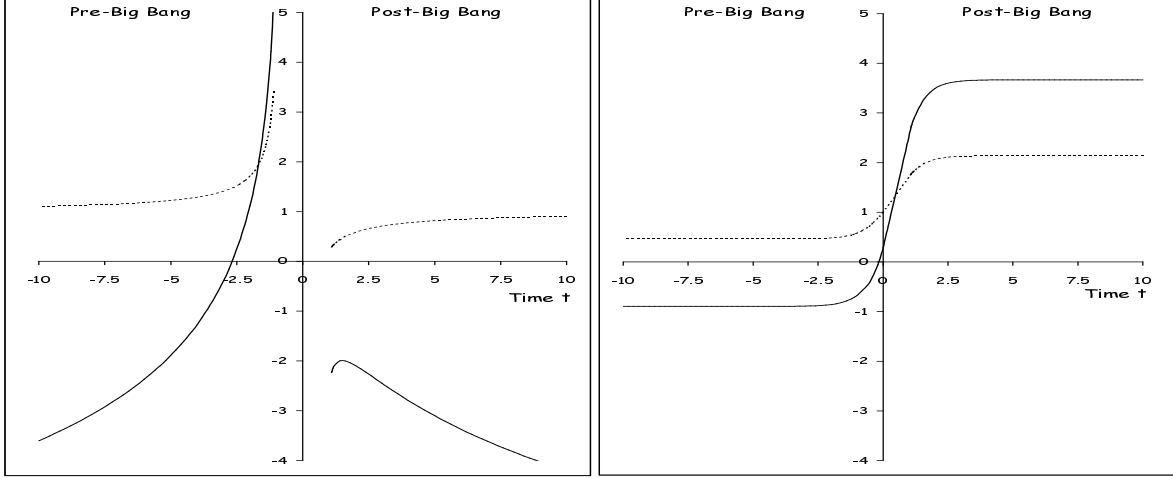


Figure 1: The solution without the cosmological constant is plotted on the left graph whereas the one with a negative cosmological constant is given by the right graph. The dashed line is associated to the scale factor and the solid line gives the scalar field behavior. For a vanishing cosmological constant, there is a forbidden region around $t = 0$ whereas a nonvanishing cosmological constant allows a continuous transition between the pre- and the post-Big Bang solutions without any initial singularity. For these plots, we took $a_0 = 1$, $\alpha_0 = 0.5$, $A = 1$.

3 Non-minimally coupling with matter

In this section, we consider quintessence models deriving from the following action

$$S = \frac{1}{2\kappa} \int \left(R - \frac{1}{2} [\nabla \phi]^2 - V(\phi) \right) \sqrt{-g} d^4x + \int f(\phi) L_m \sqrt{-g} d^4x \quad (19)$$

where the scalar field is assumed to be minimally coupled with the gravitation and in interaction with the matter field. For $f(\phi) = e^{2\phi}$, this action reduces to the string effective action studied in the previous section but written in the Einstein frame, i.e. after the following conformal transformation $g_{\alpha\beta} \rightarrow e^\phi g_{\alpha\beta}$.

The case without the potential and with $f(\phi) = e^{2\phi}$ has already been studied in [6]. If we add a potential $V(\phi)$ and if we consider a more general coupling $f(\phi)$, we can

generalize the duality transformation found in [6] as follows

$$\left\{ \begin{array}{lll} G \rightarrow \bar{G} & = & q e^{-2\phi} G^{-1} \\ \phi \rightarrow \bar{\phi} & = & \phi - \ln q \\ g_{00} \rightarrow \bar{g}_{00} & = & q g_{00} \\ \rho_m \rightarrow f(\bar{\phi}) \bar{\rho}_m & = & q^{-1} f(\phi) \rho_m \\ p_m \rightarrow f(\bar{\phi}) \bar{p}_m & = & -q^{-1} f(\phi) p_m \\ V(\phi) \rightarrow \bar{V}(\bar{\phi}) & = & q^{-1} V(\phi) \end{array} \right. \quad (20)$$

where $q = e^{3\phi} \mid \det \tilde{G} \mid$ and where we have assumed $\bar{f}(\bar{\phi}) = f(\bar{\phi})$ in order to keep the same coupling form after duality. As earlier, the equation of state of the matter field remains the same under duality only for dust ($p_m = 0$).

The transformation on $V(\phi)$ given in (20) is necessary but not sufficient. Indeed it does not imply that the dual and the initial potentials are the same function of the scalar field, i.e. $\bar{V}(\bar{\phi}) = V(\bar{\phi})$. It is straightforward to notice that the potential form is maintained invariant only with an exponential potential given by $V(\phi) = \Lambda e^\phi$. Then we have:

$$\bar{V}(\bar{\phi}) = V(\bar{\phi}) = \Lambda e^{\bar{\phi}} = \Lambda q^{-1} e^\phi = q^{-1} V(\phi) \quad (21)$$

Furthermore, if we want the dual solution to be written in the cosmic time, i.e. if we want $g_{00} \rightarrow \bar{g}_{00} = g_{00} = -1$, it is necessary to add the following transformation on the timelike coordinate

$$t \rightarrow \bar{t}(t) = \int \sqrt{q(t)} dt \quad (22)$$

When the background is given by a spatially flat FLRW metric ($g_{00} = -1$), the field equations derived from the action (19) with $V(\phi) = \Lambda e^\phi$ and $f(\phi) = e^{2\phi}$ can be written as

$$3 \frac{a'^2}{a^2} = \frac{1}{4} \phi'^2 + \frac{1}{2} \Lambda e^\phi + e^{2\phi} \kappa \rho_m \quad (23)$$

$$2 \frac{a''}{a} + \frac{a'^2}{a^2} = -\frac{1}{4} \phi'^2 + \frac{1}{2} \Lambda e^\phi - e^{2\phi} \kappa p_m \quad (24)$$

$$\phi'' + 3 \phi' \frac{a'}{a} = -\Lambda e^\phi - e^{2\phi} \kappa [\rho_m - 3 p_m] \quad (25)$$

We obtain the following peculiar solution of this system:

$$a(t) \propto \sqrt{\Lambda t^2 / 4} \quad \phi(t) = -\ln [\Lambda t^2 / 4] \quad p_m = \rho_m = 0 \quad (26)$$

This solution is “self-dual”, i.e. it is invariant under the transformations given by (20)-(22). Figure 2 shows the behavior of this solution in the pre- and post-Big Bang phases with $\Lambda = 1$. It has to be noted that the initial singularity still exists at $t = 0$.

One way to avoid the additional timelike transformation given by (22) on the dual solution is to work with the conformal time defined by:

$$d\tau = \pm a^{-3} dt = \pm \bar{a}^{-3} d\bar{t} = d\bar{\tau} \quad (27)$$

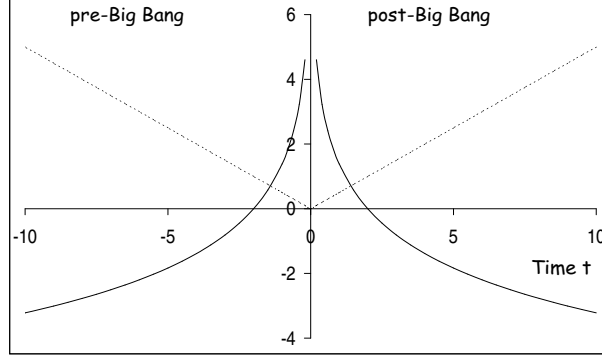


Figure 2: The self-dual solution given by (26) with $\Lambda = 1$ has been used for the pre- and for the post-Big Bang phases. The dashed line shows the scale factor behavior whereas the solid line represents the scalar field.

If we apply the conformal transformation $a \rightarrow a e^{-\phi/2}$ on the solution given by (10)-(12), we get another solution of equations (23)-(25) written in the conformal time:

$$a(\tau) = \frac{a_0 e^{-\alpha_0 \tau / 2}}{\sqrt{C_1 [\cosh(\sqrt{3} \alpha_0 \tau) - \delta_0]}} \quad (28)$$

$$\kappa \rho(\tau) = A a_0^{-3} e^{-3 \alpha_0 \tau} \quad (29)$$

$$\phi(\tau) = 3 \alpha_0 \tau + \ln \left(C_1 [\cosh(\sqrt{3} \alpha_0 \tau) - \delta_0] \right) \quad (30)$$

with the constraints (13). In this case, the conformal time is related to the cosmic time by the relation (27). We have not been able to find an analytical expression for the function $\tau(t)$. Using (20), we obtain the following dual solution:

$$\bar{a}(\tau) = \frac{a_0^2 e^{\alpha_0 \tau / 2}}{\sqrt{C_1 [\cosh(\sqrt{3} \alpha_0 \tau) - \delta_0]}} \quad (31)$$

$$\kappa \bar{\rho}(\tau) = A a_0^3 e^{3 \alpha_0 \tau} \quad (32)$$

$$\bar{\phi}(\tau) = -3 \alpha_0 \tau - 6 \ln(a_0) + \ln \left(C_1 [\cosh(\sqrt{3} \alpha_0 \tau) - \delta_0] \right) \quad (33)$$

On the figure 3, we plot this solution (28)-(30) and its dual (31)-(33) with $a_0 = 1$. For comparison, the solution is represented with and without the scalar potential. The relation between t and τ has been numerically computed. Again, for $a_0 = 1$ and in presence of the potential, the solution and its dual can be smoothly connected, without any singularity.

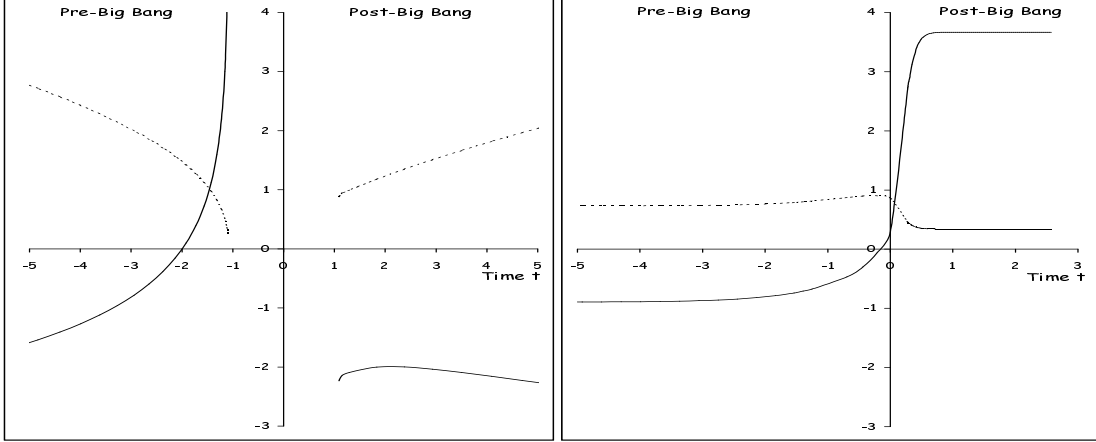


Figure 3: The solutions given by (28)-(30) and (31)-(33) are plotted on this figure. The dashed line gives the scale factor behavior whereas the solid line stands for the scalar field. The case without the potential ($\delta_0 = 1$) stands on the left graph whereas the one with a potential ($\delta_0 = 0.5$) is given on the right. The use of a non-vanishing scalar potential allows a smooth transition between the pre- and the post-Big Bang solutions. For these plots, we took $a_0 = 1$, $\alpha_0 = 0.5$, $A = 1$.

4 Minimally coupled scalar field

Now we turn our attention to the quintessence models based on a minimally coupled scalar field. These are of special interest because they are the most invoqued in the framework of the supernovae observations. These models are derived from the following action²:

$$S = \frac{1}{2\kappa} \int \left[R - \frac{1}{2} [\nabla \phi]^2 - V(\phi) \right] \sqrt{-g} d^4x + \int L_m \sqrt{-g} d^4x \quad (34)$$

Taking into account what has been done in the previous section, it is easy to show that the transformation given by

$$\begin{aligned} G &\rightarrow \bar{G} = q e^{-2\phi} G^{-1} \\ \phi &\rightarrow \bar{\phi} = \phi - \ln q \\ g_{00} &\rightarrow \bar{g}_{00} = q g_{00} \\ V(\phi) = \Lambda e^{\phi} &\rightarrow \bar{V}(\bar{\phi}) = q^{-1} V(\phi) = \Lambda e^{\bar{\phi}} \\ L_m &\rightarrow \bar{L}_m = q^{-1} L_m \end{aligned} \quad (35)$$

²The terms associated to the scalar field usually appear in the action in another way, i.e. $R - [\nabla \phi]^2 - 2V(\phi)$. We do not use these conventional notations and keep the notations already used in [6] so that no factor $\sqrt{2}$ appears in the duality transformations.

with $q = e^{3\phi} \mid \det G \mid$ does not modify the action (34). The field equations deriving from action (34) with $V(\phi) = \Lambda e^\phi$ and $g_{00} = -1$ can be written as:

$$3 \frac{a'^2}{a^2} = \frac{1}{4} \phi'^2 + \frac{1}{2} \Lambda e^\phi + \kappa \rho_m \quad (36)$$

$$2 \frac{a''}{a} + \frac{a'^2}{a^2} = -\frac{1}{4} \phi'^2 + \frac{1}{2} \Lambda e^\phi - \kappa p_m \quad (37)$$

$$\phi'' + 3 \phi' \frac{a'}{a} = -\Lambda e^\phi \quad (38)$$

One can easily check that equations (36)-(38) remain invariant under the transformation given by (35) only if the matter field changes as

$$\begin{cases} \rho_m \rightarrow \bar{\rho}_m = q^{-1} \rho_m \\ p_m \rightarrow \bar{p}_m = q^{-1} [2p_m - \rho_m] \end{cases} \quad (39)$$

which means

$$w_m = \frac{p_m}{\rho_m} \rightarrow \bar{w}_m = \frac{\bar{p}_m}{\bar{\rho}_m} = 2w_m - 1 \quad (40)$$

In this case, the equation of state of the matter field is invariant under the scale factor duality only for a stiff fluid ($w_m = 1$). On the other hand, an initial dust fluid ($w_m = 0$) becomes a cosmological constant ($\bar{w}_m = -1$) after duality. Note that, as in the previous case, if we want the dual solution to be written in the cosmic time, the duality transformation given by (35) and (39) must be followed by the transformation (22) on the timelike coordinate.

5 Conclusions

The scale factor duality is a symmetry property of the scalar-tensor theories which has already been studied in the literature. In this paper, we have expanded those studies to quintessence models with a scalar potential. The introduction of this potential has been dictated by the recent supernovae observations which suggested a currently accelerating universe. Three kinds of cosmological models have been considered in this paper: one with a coupling between the scalar field and gravitation, one allowing an interaction between the scalar component and the matter field and the last one with a minimally coupled quintessence field. In each case, the potential form which allows the existence of a scale factor duality symmetry has been found.

In the framework of scalar-tensor theory where the scalar field is coupled with gravitation, the potential form depends on the coupling, i.e. on the gravitation theory. In the case of the effective string theory, only a theory with constant scalar potential allows a scale factor duality. In the case of a Brans-Dicke theory, the potential has to be linear whereas in a scalar-tensor theory, the potential must have a power-law form. Moreover, only the equation of state of a dust fluid is invariant under this symmetry when the scalar field is not coupled with the matter field.

When there is no coupling between the scalar field and gravitation, the potential must have an exponential form. In order to have an invariant equation of state, the matter field must be dust, if interactions between the scalar and matter components are allowed, and stiff, if the scalar field is self-interacting.

Last but not least, we have also shown that in some cases, the scalar potential allows a smooth transition between the pre- and the post-Big Bang solutions which was not possible without such a potential. This is perhaps the most important point. Indeed, till today, no satisfying model can explain the transition between the pre- and the post-Big Bang phases. It would be an interesting result if the potential necessary in the context of quintessence would also permit to carry out this smooth transition: we have shown that this is quite likely.

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